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TWO-CENTER PROBLEM ORBITS AS INTERMEDIATE ORBITS FOR THE RESTRICTED THREE-BODY PROBLEM

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SUMMARY

The restricted three-body problem is treated by keeping for the intermediate orbit Hamiltonian all terms except those representing a generalization of the centrifugal and Coriolis forces. The intermediate orbit, therefore, takes into account the gravitational forces of both finite masses. An approximation is presented for representing a class of intermediate orbits similar to the orbits suggested for the Apollo project. The perturbation equations are given in outline.

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INTRODUCTION

The early masters in the field of celestial mechanics, particularly Euler, discovered that the problem of the motion of a particle about two fixed mass points ("two fixed centra") can be reduced to quadratures, although these integrals do not represent elementary functions (Reference 1, p. 145). Over a hundred years later Charlier (Reference 2, Vol. II) suggested that the two-center orbits could be used as intermediate orbits in the restricted three-body problem, and he sketched a broad outline for such a procedure. His program was carried out in some detail by Samter (Reference 3). He found the calculations lengthy and the perturbations relatively large, which seemed to dampen his enthusiasm somewhat for Charlier's idea. Samter treated the case of the two given masses' being nearly equal, basing his approximations on this circumstance. In the present article we investigate the efficacy of Charlier's method for orbits of the type suggested for the Apollo mission. The near-ejection type orbits admit of an approximate representation for the intermediate orbits in terms of elementary functions.

THE EQUATIONS OF MOTION

Let the two finite masses in the restricted three-body problem be indicated by μ and $1 - \mu$. These two masses are assumed to be describing circular orbits about their common center of gravity. Let (x, y) be barycentric coordinates for a system rotating with the masses so that the coordinates of the particle with mass μ are $(1 - \mu, 0)$ and of the particle with mass $1 - \mu$ are $(-\mu, 0)$. The Lagrangian and force function for the restricted three-body problem in this rotating coordinate system then are (Reference 1, p. 350):

$$\left.\begin{aligned} L &= \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + (x\dot{y} - y\dot{x}) + U(x, y) \quad , \\ U &= \frac{1}{2}\left(x^2 + y^2\right) + \frac{1 - \mu}{\left|(x + \mu)^2 + y^2\right|^{1/2}} + \frac{\mu}{\left|(x - 1 + \mu)^2 + y^2\right|^{1/2}} \quad . \end{aligned}\right\}$$

Since the two-center problem is separable in either elliptic or bipolar coordinates and since we shall be using (certain generalizations of) two-center orbits as intermediate orbits we determine the Lagrangian, force function, and Hamiltonian in these new coordinates.

Bipolar coordinates, r_1, r_2, are the distances of (x, y) from the points $(-\mu, 0)$ and $(1 - \mu, 0)$, respectively. Thus

$$\left.\begin{aligned} r_1 &= \left|(x+\mu)^2 + y^2\right|^{1/2} , \\ r_2 &= \left|(x-1+\mu)^2 + y^2\right|^{1/2} . \end{aligned}\right\}$$

Elliptic coordinates, ξ, η, have simple representations in terms of r_1, r_2:

$$\left.\begin{aligned} \cos\xi &= r_1 - r_2 , \\ \cosh\eta &= r_1 + r_2 . \end{aligned}\right\}$$

Consequently, the transformation formulae from the barycentric coordinates to the elliptic are

$$\left.\begin{aligned} x &= -\mu + \frac{1}{2} + \frac{1}{2}\cos\xi\cosh\eta , \\ y &= \frac{1}{2}\sin\xi\sinh\eta . \end{aligned}\right\}$$

The restricted three-body problem Hamiltonian in bipolar coordinates has the form

$$\begin{aligned} H = {} & \frac{1}{2}\hat{\mathfrak{p}}_1^2 - \frac{1 - r_1^2 - r_2^2}{2r_1 r_2}\hat{\mathfrak{p}}_1\hat{\mathfrak{p}}_2 + \frac{1}{2}\hat{\mathfrak{p}}_2^2 + \frac{\mu r_2\sqrt{\left[(r_1+r_2)^2 - 1\right]\left[1 - (r_1 - r_2)^2\right]}}{2r_1 r_2}\hat{\mathfrak{p}}_1 \\ & - \frac{(1-\mu) r_1\sqrt{\left[(r_1+r_2)^2 - 1\right]\left[1 - (r_1 - r_2)^2\right]}}{2r_1 r_2}\hat{\mathfrak{p}}_2 - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} , \end{aligned} \tag{1}$$

where $\hat{\mathfrak{p}}_i$ are the momenta conjugate to r_i. And in the case of elliptic coordinates with $\mathfrak{p}_1$ conjugate to ξ and $\mathfrak{p}_2$ conjugate to η the restricted three-body Hamiltonian is

$$\begin{aligned} H = {} & \frac{2}{\cosh^2\eta - \cos^2\xi}\mathfrak{p}_1^2 + \frac{2}{\cosh^2\eta - \cos^2\xi}\mathfrak{p}_2^2 - \frac{\sinh\eta\left[\cosh\eta + (1-2\mu)\cos\xi\right]}{\cosh^2\eta - \cos^2\xi}\mathfrak{p}_1 \\ & - \frac{\sin\xi\left[(1-2\mu)\cosh\eta + \cos\xi\right]}{\cosh^2\eta - \cos^2\xi}\mathfrak{p}_2 - \frac{2(1-\mu)}{\cosh\eta + \cos\xi} - \frac{2\mu}{\cosh\eta - \cos\xi} . \end{aligned} \tag{2}$$

The formulae for the two-center problem are

$$L^{(2)} = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + U^{(2)} = \frac{1}{8}\left(\cosh^2\eta - \cos^2\xi\right)\left(\dot{\xi}^2 + \dot{\eta}^2\right) + U^{(2)} , \tag{3a}$$

$$U^{(2)} = \frac{1-\mu}{\left|(x+\mu)^2 + y^2\right|^{1/2}} + \frac{\mu}{\left|(x-1+\mu)^2 + y^2\right|^{1/2}} = \frac{2(1-\mu)}{\cosh\eta + \cos\xi} + \frac{2\mu}{\cosh\eta - \cos\xi} , \tag{3b}$$

$$H^{(2)} = \frac{2}{\cosh^2\eta - \cos^2\xi}\, p_1^2 + \frac{2}{\cosh^2\eta - \cos^2\xi}\, p_2^2 - \frac{2(1-\mu)}{\cosh\eta + \cos\xi} - \frac{2\mu}{\cosh\eta - \cos\xi} . \tag{3c}$$

Actually, of course, Equations 3a, 3b, and 3c are not strictly the formulae for the two fixed centers problem if we adhere to our definition of x, y since these represent coordinates in a rotating system whereas in the Newtonian two-center problem the two masses are fixed in an inertial frame. A more nearly precise statement would be that $H^{(2)}$ represents the H of (2) with the Coriolis and centrifugal force terms (the terms linear in p_i) suppressed. Note, though, that the p_i represent different quantities in H and $H^{(2)}$ since the Lagrangian for the restricted three-body problem has terms in $\dot{\xi}$ and $\dot{\eta}$ as well as $\dot{\xi}^2$ and $\dot{\eta}^2$.

PARTITION AND SEPARATION OF VARIABLES

In view of (2) we see that Hamilton's Equation for the restricted three-body problem in elliptic coordinates is

$$\frac{\partial W}{\partial t} + \left(\cosh^2\eta - \cos^2\xi\right)^{-1}\left\{2\left(\frac{\partial W}{\partial \xi}\right)^2 + 2\left(\frac{\partial W}{\partial \eta}\right)^2 - \sinh\eta\left[\cosh\eta + (1-2\mu)\cos\xi\right]\frac{\partial W}{\partial \xi}\right.$$
$$\left. - \sin\xi\left[(1-2\mu)\cosh\eta + \cos\xi\right]\frac{\partial W}{\partial \eta} - 2\left[\cosh\eta - (1-2\mu)\cos\xi\right]\right\} = 0 . \tag{4}$$

The variables ξ, η do not separate in this partial differential equation although they do in Hamilton's Equation formed from $H^{(2)}$. Thus, a separable problem for an intermediate orbit would result if H be partitioned into $H_0 + H_1$ with $H_0 = H^{(2)}$. However, a larger class of intermediate orbits is obtained from the following setup:

$$\left\{\begin{aligned} H &= H_0 + H_1 , \\ H_0 &= 2\left(\cosh^2\eta - \cos^2\xi\right)^{-1}\left(p_1^2 + p_2^2\right) - \left(\cosh^2\eta - \cos^2\xi\right)^{-1}\left[u(\xi)\, p_1 + v(\eta)\, p_2\right] \\ &\qquad - \frac{2(1-\mu)}{\cosh\eta + \cos\xi} - \frac{2\mu}{\cosh\eta - \cos\xi} , \\ H_1 &= -\left(\cosh^2\eta - \cos^2\xi\right)^{-1}\left\{\left(\sinh\eta\cdot\left[\cosh\eta + (1-2\mu)\cos\xi\right] - u(\xi)\right)p_1 + \left(\sin\xi\cdot\left[(1-2\mu)\cosh\eta + \cos\xi\right] - v(\eta)\right)p_2\right\}. \end{aligned}\right. \tag{5}$$

The functions $u(\xi)$, $v(\eta)$ are at our disposal. If they are taken as zero then H_0 is formally the same as the two-center Hamiltonian and consequently is separable. However, the more general H_0 given in Equation 5 is still separable and we have the functions u, v available to be chosen for our convenience, say to minimize certain perturbations, for example.

Somewhat simpler expressions ultimately result if a contact transformation is made on H which effectively completes the square in the p_1 and p_2 polynomials in H_0, the transformation being

$$\left.\begin{aligned} \xi &= \lambda_1 , \\ \eta &= \lambda_2 , \end{aligned}\right\} \qquad \left.\begin{aligned} p_1 &= \Lambda_1 + \frac{1}{2} u(\lambda_1) , \\ p_2 &= \Lambda_2 + \frac{1}{2} v(\lambda_2) . \end{aligned}\right\} \tag{6}$$

Then

$$\left.\begin{aligned} H &= \widetilde{H}_0 + \widetilde{H}_1 , \\ \widetilde{H}_0 &= 2\left(\cosh^2 \lambda_2 - \cos^2 \lambda_1\right)^{-1} \left[\Lambda_1^2 + \Lambda_2^2 - \frac{1}{4} u^2(\lambda_1) + (1 - 2\mu) \cos \lambda_1 - \frac{1}{4} v^2(\lambda_2) - \cosh \lambda_2\right] , \\ \widetilde{H}_1 &= -\left(\cosh^2 \lambda_2 - \cos^2 \lambda_1\right)^{-1} \Bigg(\Big\{\sinh \lambda_2 \cdot \left[\cosh \lambda_2 + (1 - 2\mu) \cos \lambda_1\right] - u(\lambda_1)\Big\} \left\{\Lambda_1 + \frac{1}{2} u(\lambda_1)\right\} \\ &\qquad + \Big\{\sin \lambda_1 \cdot \left[(1 - 2\mu) \cosh \lambda_2 + \cos \lambda_1\right] - v(\lambda_2)\Big\} \left\{\Lambda_2 + \frac{1}{2} v(\lambda_2)\right\}\Bigg) . \end{aligned}\right\} \tag{7}$$

Thus the Hamilton Equation for the intermediate orbit is

$$\frac{\partial \widetilde{W}_0}{\partial t} + 2\left(\cosh^2 \lambda_2 - \cos^2 \lambda_1\right)^{-1} \left[\left(\frac{\partial \widetilde{W}_0}{\partial \lambda_1}\right)^2 + \left(\frac{\partial \widetilde{W}_0}{\partial \lambda_2}\right)^2 - \frac{1}{4} u^2(\lambda_1) + (1 - 2\mu) \cos \lambda_1 - \frac{1}{4} v^2(\lambda_2) - \cosh \lambda_2\right] = 0 . \tag{8}$$

By employing the separation of variables method we obtain as a complete integral

$$\widetilde{W}_0 = -\hbar t + \int^{\lambda_1} \pm\sqrt{\frac{1}{4} u^2(\lambda) - (1 - 2\mu) \cos \lambda - \frac{1}{2} \hbar \cos^2 \lambda - \alpha}\, d\lambda + \int^{\lambda_2} \pm\sqrt{\frac{1}{4} v^2(\lambda) + \cosh \lambda + \frac{1}{2} \hbar \cosh^2 \lambda + \alpha}\, d\lambda \tag{9}$$

wherein $\hbar$ and α are canonical constants.

THE INTERMEDIATE ORBIT

According to the Hamilton-Jacobi theory the intermediate orbit is provided from Equation 9 by

$$
\left.
\begin{aligned}
t - \beta_1 &= \frac{\partial \widetilde{W}_0}{\partial \hbar} , \\
-\beta_2 &= \frac{\partial \widetilde{W}_0}{\partial \alpha} , \\
\Lambda_1 &= \frac{\partial \widetilde{W}_0}{\partial \lambda_1} , \\
\Lambda_2 &= \frac{\partial \widetilde{W}_0}{\partial \lambda_2} ,
\end{aligned}
\right\} \tag{10}
$$

where β_1, β_2 are the remaining two canonical constants.

In order to obtain a more specific class of orbits we shall specialize Equation 9 to some extent by taking the functions u and v to be of the form

$$
\left.
\begin{aligned}
\frac{1}{4} u^2 (\lambda) &\equiv u_0 + u_1 \cos \lambda + u_2 \cos^2 \lambda , \\
\frac{1}{4} v^2 (\lambda) &\equiv v_0 + v_1 \cosh \lambda + v_2 \cosh^2 \lambda ,
\end{aligned}
\right\} \tag{11}
$$

with u_i, v_i constants. This choice keeps the orbits within the general types that appear in the two-center problem.

Introduce new constants μ_i, ν_i defined by

$$
\left.
\begin{aligned}
\mu_0 &= u_0 - \alpha , & \nu_0 &= v_0 + \alpha , \\
\mu_1 &= u_1 - (1 - 2\mu) , & \nu_1 &= v_1 + 1 , \\
\mu_2 &= u_2 - \frac{1}{2} \hbar , & \nu_2 &= v_2 + \frac{1}{2} \hbar .
\end{aligned}
\right\} \tag{12}
$$

From Equations 9 to 12 we then obtain the equations defining the intermediate orbit:

$$
t - \beta_1 = -\frac{1}{4} \int_{\lambda_1^{(0)}}^{\lambda_1} \frac{\cos^2 \lambda \, d\lambda}{\pm \sqrt{\mu_0 + \mu_1 \cos \lambda + \mu_2 \cos^2 \lambda}} + \frac{1}{4} \int_{\lambda_2^{(0)}}^{\lambda_2} \frac{\cosh^2 \lambda \, d\lambda}{\pm \sqrt{\nu_0 + \nu_1 \cosh \lambda + \nu_2 \cosh^2 \lambda}} , \tag{13a}
$$

$$-\beta_2 = -\int_{\lambda_1^{(0)}}^{\lambda_1} \frac{d\lambda}{\pm\sqrt{\mu_0 + \mu_1 \cos\lambda + \mu_2 \cos^2\lambda}} + \int_{\lambda_2^{(0)}}^{\lambda_2} \frac{d\lambda}{\pm\sqrt{\nu_0 + \nu_1 \cosh\lambda + \nu_2 \cosh^2\lambda}} , \tag{13b}$$

$$\Lambda_1 = \pm\sqrt{\mu_0 + \mu_1 \cos\lambda_1 + \mu_2 \cos^2\lambda_1} , \tag{14a}$$

$$\Lambda_2 = \pm\sqrt{\nu_0 + \nu_1 \cosh\lambda_2 + \nu_2 \cosh^2\lambda_2} . \tag{14b}$$

We defer making the choice for the constants $\lambda_i^{(0)}$ in Equations 13a and 13b until we have examined the character of the integrals more fully.

The integrals are elliptic and cannot be expressed in compact form in terms of elementary functions apart from very exceptional values of the constants μ_i, ν_i. The general character of the orbits depends most immediately on the zeros of the polynomials $\mu_2 x^2 + \mu_1 x + \mu_0$ and $\nu_2 x^2 + \nu_1 x + \nu_0$ inasmuch as the vanishing of these polynomials provides the points of zero momenta by Equations 14a and 14b. The books by Charlier and Wintner (References 1 and 2) discuss the dynamical significance of these roots in some detail for rather general problems. A proposed orbit for the Apollo capsule is frequently described rather loosely as being first an arc of a two-body ellipse extending from near the earth to the neutral point between the earth and the moon and then continuing on another two-body ellipse to near the moon, thereby describing a very slender figure of the form:  Now, orbits of this general character are included in Equation 13a and 13b: these are the cases I$b\alpha$ and I$b\beta$ of Charlier (Reference 2, Vol. I, p. 126) which represent quasi-lemniscates winding about the two mass points. We shall assume

$$\left.\begin{aligned} \mu_2 &> 0 , \\ \nu_2 &< 0 , \end{aligned}\right\}$$

which is the case, for example, if u_2, v_2 are zero or are quite small, and $h < 0$. Let ρ_1, ρ_2 be the roots of $\mu_2 x^2 + \mu_1 x + \mu_0 = 0$ and let r_1, r_2 be the roots of $\nu_2 x^2 + \nu_1 x + \nu_0 = 0$. To give the orbits in question, have r_1, r_2 real with $r_1 > 1 > r_2$ and ρ_1, ρ_2 either complex or else real with $\rho_1 \geq \rho_2 > 1$. This means that the ranges of the variables λ_1, λ_2 are $0 \leq \lambda_1 \leq 2\pi$, $0 \leq \cosh\lambda_2 \leq r_1$, since $\mu_2 \cos^2\lambda_1 + \mu_1 \cos\lambda_1 + \mu_0 = \mu_2 (\cos\lambda_1 - \rho_1)(\cos\lambda_1 - \rho_2)$ does not now vanish or go negative for real λ_1, and $\nu_2 \cosh^2\lambda_2 + \nu_1 \cosh\lambda_2 + \nu_0$ remains positive or zero only in indicated range for λ_2. The explicit formulae for the roots in terms of the coefficients are

$$\left.\begin{aligned} \rho_1 &= \frac{-\mu_1 + \sqrt{\mu_1^2 - 4\mu_0\mu_2}}{2\mu_2} , & r_1 &= \frac{-\nu_1 - \sqrt{\nu_1^2 - 4\nu_0\nu_2}}{2\nu_2} , \\ \rho_2 &= \frac{-\mu_1 - \sqrt{\mu_1^2 - 4\mu_0\mu_2}}{2\mu_2} , & r_2 &= \frac{-\nu_1 + \sqrt{\nu_1^2 - 4\nu_0\nu_2}}{2\nu_2} . \end{aligned}\right\} \tag{15}$$

It appears from the foregoing that a rather convenient choice for $\lambda_1^{(0)}$ and $\lambda_2^{(0)}$ in Equations 13a and 13b is $\lambda_1^{(0)} = 0$, $\lambda_2^{(0)} = -\hat{r}_1$ where $\cosh \hat{r}_1 = r_1$. The integrals in these equations thereupon vanish at the point (in barycentric coordinates)

$$x = -\mu + \frac{1}{2} + \frac{1}{2} r_1 , \qquad y = 0 ,$$

i.e., the point on the line connecting the two mass points, on the outside of the point with mass μ. Thus β_1 is the instant of passage through this point.

The solving of Equations 13a and 13b for λ_1, λ_2 in terms of t can be effected in terms of elliptic functions but the expressions are prolix. Simpler formulae can be obtained through the introduction of parameters σ_1, σ_2 defined as

$$\sigma_1 = \int_0^{\lambda_1} \frac{d\lambda}{\pm \sqrt{\mu_0 + \mu_1 \cos \lambda + \mu_2 \cos^2 \lambda}} ,$$
$$\sigma_2 = \int_{-\hat{r}_1}^{\lambda_2} \frac{d\lambda}{\pm \sqrt{\nu_0 + \nu_1 \cosh \lambda + \nu_2 \cosh^2 \lambda}} . \tag{16}$$

These are not independent, of course, since we have from Equations 13a and 13b,

$$\sigma_2 = \sigma_1 - \beta_2 .$$

The integrals in Equations 16 have relatively simple expressions in terms of inverse elliptic functions, and the equations solved for λ_1 and λ_2 give (Reference 4, p. 134):

$$\cos \lambda_1 = \frac{(\rho_2 + 1) - 2\rho_2 \, \mathrm{sn}^2 \left[\frac{1}{2} \sqrt{\mu_2 (\rho_1 - 1)(\rho_2 + 1)} \, \sigma_1 , k_1\right]}{(\rho_2 + 1) - 2 \, \mathrm{sn}^2 \left[\frac{1}{2} \sqrt{\mu_2 (\rho_1 - 1)(\rho_2 + 1)} \, \sigma_1 , k_1\right]} , \qquad k_1 = \sqrt{\frac{2(\rho_1 - \rho_2)}{(\rho_1 - 1)(\rho_2 + 1)}} , \tag{17a}$$

$$\cosh \lambda_2 = \frac{2r_1 - (r_1 - 1) \, \mathrm{sn}^2 \left[\frac{1}{2} \sqrt{-2\nu_2 (r_1 - r_2)} \, (\sigma_1 - \beta_2) , k_2\right]}{2 + (r_1 - 1) \, \mathrm{sn}^2 \left[\frac{1}{2} \sqrt{-2\nu_2 (r_1 - r_2)} \, (\sigma_1 - \beta_2) , k_2\right]} , \qquad k_2 = \sqrt{\frac{(r_1 - 1)(r_2 + 1)}{2(r_1 - r_2)}} , \quad r_2 > -1 , \tag{17b}$$

$$= \frac{r_1 (1 - r_2) + r_2 (r_1 - 1) \, \mathrm{sn}^2 \left[\frac{1}{2} \sqrt{-\nu_2 (r_1 + 1)(1 - r_2)} \, (\sigma_1 - \beta_2) , k_2\right]}{(1 - r_2) + (r_1 - 1) \, \mathrm{sn}^2 \left[\frac{1}{2} \sqrt{-\nu_2 (r_1 + 1)(1 - r_2)} \, (\sigma_1 - \beta_2) , k_2\right]} , \qquad k_2 = \sqrt{\frac{(r_1 - 1)(-1 - r_2)}{(r_1 + 1)(1 - r_2)}} , \quad r_2 < -1 .$$

The function $\mathrm{sn}(u, k)$ is the Jacobi elliptic function, the sine amplitude of u with parameter k. The formula for $\cos \lambda_1$ has all quantities pure real if we restrict ourselves to the case of ρ_1, ρ_2

pure real, as we shall do from here on. Both formulae for $\cosh\lambda_2$ are valid no matter whether $r_2 > -1$ or $r_2 < -1$, but the form of the first is such that all quantities in it are real if $r_2 > -1$, and similarly for the second if $r_2 < -1$. Hereafter, though, we shall use only the first expression for $\cosh\lambda_2$.

Equations 17a and 17b represent a pair of parametric equations with parameter s_1 for the intermediate orbit. If these expressions are introduced into Equations 14a and 14b formulae for the momenta in terms of s_1 result:

$$\Lambda_1 = \frac{2a_1\sqrt{\rho_2^2 - 1}\,\mathrm{dn}(a_1 s_1, k_1)}{(\rho_2 + 1) - 2\,\mathrm{sn}^2(a_1 s_1, k_1)}, \qquad a_1 = \frac{1}{2}\sqrt{\mu_2(\rho_1 - 1)(\rho_2 + 1)}, \qquad k_1 = \sqrt{\frac{2(\rho_1 - \rho_2)}{(\rho_1 - 1)(\rho_2 + 1)}}, \tag{18a}$$

$$\Lambda_2 = \frac{2a_2\sqrt{r_1^2 - 1}\,\mathrm{sn}(a_2 s_2, k_2)\,\mathrm{dn}(a_2 s_2, k_2)}{2 + (r_1 - 1)\,\mathrm{sn}^2(a_2 s_2, k_2)}, \qquad a_2 = \frac{1}{2}\sqrt{-2\nu_2(r_1 - r_2)}, \qquad k_2 = \sqrt{\frac{(r_1 - 1)(r_2 + 1)}{2(r_1 - r_2)}},$$

$$s_2 = s_1 - \beta_2 . \tag{18b}$$

By using the counterpart for elliptic functions of the trigonometric half-angle formulae (Reference 4, p. 146) we can express $\cos\lambda_1$ and $\cosh\lambda_2$ in a form devoid of squared elliptic functions:

$$\cos\lambda_1 = \frac{-\rho_2 + 1 + 2\rho_2\,\mathrm{cn}(2a_1 s_1, k_1) + (\rho_2 + 1)\,\mathrm{dn}(2a_1 s_1, k_1)}{\rho_2 - 1 + 2\,\mathrm{cn}(2a_1 s_1, k_1) + (\rho_2 + 1)\,\mathrm{dn}(2a_1 s_1, k_1)}, \quad \begin{cases} a_1 = \frac{1}{2}\sqrt{\mu_2(\rho_1 - 1)(\rho_2 + 1)}, \\ k_1 = \sqrt{\dfrac{2(\rho_1 - \rho_2)}{(\rho_1 - 1)(\rho_2 + 1)}}, \end{cases} \tag{19a}$$

$$\cosh\lambda_2 = \frac{r_1 + 1 + (r_1 - 1)\,\mathrm{cn}(2a_2 s_2, k_2) + 2r_1\,\mathrm{dn}(2a_2 s_2, k_2)}{r_1 + 1 - (r_1 - 1)\,\mathrm{cn}(2a_2 s_2, k_2) + 2\,\mathrm{dn}(2a_2 s_2, k_2)}, \quad \begin{cases} a_2 = \frac{1}{2}\sqrt{-2\nu_2(r_1 - r_2)}, \\ k_2 = \sqrt{\frac{1}{2}(r_1 - 1)(r_2 + 1)(r_1 - r_2)^{-1}}, \\ s_2 = s_1 - \beta_2 . \end{cases} \tag{19b}$$

Observe from Equations 17 or 19 that the degenerate case resulting from letting $r_1 \to 1$ gives simply $\cosh\lambda_2 = 1$ which represents the straight line solution running from one of the mass points to the other. Also note that putting $r_1 = 1$ causes k_2 to vanish. And k_1 is small if ρ_1 is close to ρ_2. Therefore, there are lemniscate-type orbits for which the elliptic function parameters k_1 and k_2 are small, which suggests expanding $\cos\lambda_1$, $\cosh\lambda_2$, Λ_1, Λ_2 in powers of k_1 and k_2. The result is especially convenient since these expansions happen to involve elementary functions only. The

Fourier expansions of the elliptic functions (Reference 4, p. 147),

$$
\left.
\begin{aligned}
2K\mathscr{k}\,\mathrm{sn}(2Ku) &= 4\pi \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1 - q^{2n+1}} \sin(2n+1)\pi u \ , \\
2K\mathscr{k}\,\mathrm{cn}(2Ku) &= 4\pi \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1 + q^{2n+1}} \cos(2n+1)\pi u \ , \\
2K\,\mathrm{dn}(2Ku) &= \pi + 4\pi \sum_{n=1}^{\infty} \frac{q^{n}}{1 + q^{2n}} \cos(2n\pi u) \ ,
\end{aligned}
\right\} \quad (20)
$$

provide a means for obtaining representations valid for small $\mathscr{k}$ since

$$
q = \frac{k^2}{16}\left[1 + 2\left(\frac{k}{4}\right)^2 + 15\left(\frac{k}{4}\right)^4 + 150\left(\frac{k}{4}\right)^6 + \cdots\right]^4 \ . \quad (21)
$$

In Equation 20 $K = K(\mathscr{k})$ is the "complete elliptic integral of the first kind":

$$
K(\mathscr{k}) = \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-\mathscr{k}^2 u^2)}} = \frac{\pi}{2}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \mathscr{k}^2\right) \ . \quad (22)
$$

If q is replaced in Equation 20 through the use of Equation 21 there is obtained

$$
\left.
\begin{aligned}
\mathrm{sn}\, 2Ku &= \sin \pi u + \left(\frac{1}{16}\sin \pi u + \frac{1}{16}\sin 3\pi u\right)\mathscr{k}^2 + O(\mathscr{k}^4) \ , \\
\mathrm{cn}\, 2Ku &= \cos \pi u + \left(-\frac{1}{16}\cos \pi u + \frac{1}{16}\cos 3\pi u\right)\mathscr{k}^2 + O(\mathscr{k}^4) \ , \\
\mathrm{dn}\, 2Ku &= 1 + \left(-\frac{1}{4} + \frac{1}{4}\cos 2\pi u\right)\mathscr{k}^2 + O(\mathscr{k}^4) \ .
\end{aligned}
\right\} \quad (23)
$$

The formulae specifying the intermediate orbit valid for small $\mathscr{k}_1$, $\mathscr{k}_2$ are then found from Equations 19, 18, and 20 to be

$$
\cos \lambda_1 = \frac{1 + \rho_2 \cos \dfrac{\pi a_1 \vartheta_1}{K_1}}{\rho_2 + \cos \dfrac{\pi a_1 \vartheta_1}{K_1}} - \frac{\left(\rho_2^2 - 1\right)\sin^2 \dfrac{\pi a_1 \vartheta_1}{K_1}}{4\left(\rho_2 + \cos \dfrac{\pi a_1 \vartheta_1}{K_1}\right)^2}\mathscr{k}_1^2 + O\left(\mathscr{k}_1^4\right) \ , \quad (24a)
$$

$$
\cosh \lambda_2 = \frac{3r_1 + 1 + (r_1 - 1)\cos \dfrac{\pi a_2 \vartheta_2}{K_2}}{r_1 + 3 - (r_1 - 1)\cos \dfrac{\pi a_2 \vartheta_2}{K_2}} - \frac{\left(r_1^2 - 1\right)\sin^2 \dfrac{\pi a_2 \vartheta_2}{K_2}}{\left[(r_1 + 3) - (r_1 - 1)\cos \dfrac{\pi a_2 \vartheta_2}{K_2}\right]^2}\mathscr{k}_2^2 + O\left(\mathscr{k}_2^4\right) \ , \quad (24b)
$$

$$\Lambda_1 = 2a_1\sqrt{\rho_2^2-1}\left\{\frac{1}{\rho_2+\cos\frac{\pi a_1 \vartheta_1}{K_1}} + \frac{-(\rho_2-1)+(\rho_2-1)\cos\frac{\pi a_1\vartheta_1}{K_1}}{4\left(\rho_2+\cos\frac{\pi a_1\vartheta_1}{K_1}\right)^2}\,k_1^2\right\}+O\left(k_1^4\right) \quad , \tag{24c}$$

$$\Lambda_2 = 4a_2\sqrt{r_1^2-1}\,\sin\frac{\pi a_2\vartheta_2}{2K_2}\left\{\frac{1}{r_1+3-(r_1-1)\cos\frac{\pi a_2\vartheta_2}{K_2}} + \frac{-3r_1-1+(4r_1+8)\cos\frac{\pi a_2\vartheta_2}{K_2}-(r_1-1)\cos^2\frac{\pi a_2\vartheta_2}{K_2}}{8\left[r_1+3-(r_1-1)\cos\frac{\pi a_2\vartheta_2}{K_2}\right]^2}\,k_2^2\right\}+O\left(k_2^4\right) , \tag{24d}$$

$$\left.\begin{aligned} a_1 &= \frac{1}{2}\sqrt{\mu_2(\rho_1-1)(\rho_2+1)} \quad , \\ k_1 &= \sqrt{2(\rho_1-\rho_2)(\rho_1-1)^{-1}(\rho_2+1)^{-1}} \quad , \\ K_1 &= K(k_1) \quad , \end{aligned}\right\} \tag{24e}$$

$$\left.\begin{aligned} a_2 &= \frac{1}{2}\sqrt{-2\nu_2(r_1-r_2)} \quad , \\ k_2 &= \sqrt{\frac{1}{2}(r_1-1)(r_2+1)(r_1-r_2)^{-1}} \quad , \\ K_2 &= K(k_2) \quad , \end{aligned}\right\} \tag{24f}$$

where $\vartheta_2 = \vartheta_1 - \beta_2$.

Better (but more complicated) approximations than those of Equations 24a through 24f can be obtained by employing the process of "convergence improvement" on Equation 20. The success of the method depends essentially on the possibility of discovering a known Fourier series that is close to the given one. The difference series will then, in general, converge more rapidly than the original. We apply this method to the cosine amplitude functions. From Equation 20 we have

$$\begin{aligned}
\frac{Kk}{2\pi}\,\mathrm{cn}(2Ku) &= \sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1+q^{2n+1}}\cos(2n+1)\pi u \quad , \\
&= \sum_{n=0}^{\infty}\left[\frac{q^{n+1/2}}{1+q^{2n+1}} - q^{n+1/2} + q^{n+1/2}\right]\cos(2n+1)\pi u \quad , \\
&= \sum_{n=0}^{\infty} q^{n+1/2}\cos(2n+1)\pi u - \sum_{n=0}^{\infty}\frac{q^{3n+3/2}}{1+q^{2n+1}}\cos(2n+1)\pi u \quad , \\
&= \frac{(1-q)\,q^{1/2}\cos\pi u}{(1+q)^2-4q\cos^2\pi u} - \sum_{n=0}^{\infty}\frac{q^{3n+3/2}}{1+q^{2n+1}}\cos(2n+1)\pi u \quad , \\
&= \frac{(1-q)\,q^{1/2}\cos\pi u}{(1+q)^2-4q\cos^2\pi u} - \frac{q^{3/2}}{1+q} - \sum_{n=1}^{\infty}\frac{q^{3n+3/2}}{1+q^{2n+1}}\cos(2n+1)\pi u \quad , \\
&= q^{1/2}\cos\pi u\,\frac{1-q-q^2+2q^2\cos 2\pi u - q^3}{(1+q)\left(1-2q\cos 2\pi u+q^2\right)} - \sum_{n=1}^{\infty}\frac{q^{3n+3/2}}{1+q^{2n+1}}\cos(2n+1)\pi u \quad .
\end{aligned}$$

The Fourier series introduced above,

$$\sum_{n=0}^{\infty} q^{n+1/2} \cos(2n+1)\pi u \quad ,$$

was summed by employing simple operations on a known series (Reference 4, p. 213). A similar process works just as well for $\mathrm{dn}\, u$ so that we find

$$\left.\begin{aligned}
\frac{Kk}{2\pi}\,\mathrm{cn}(2Ku) &= q^{1/2}\cos\pi u\,\frac{1-q-q^2+2q^2\cos 2\pi u - q^3}{(1+q)\left(1-2q\cos 2\pi u+q^2\right)} + O\left(k^9\right) \; , \\[2ex]
\frac{2K}{\pi}\,\mathrm{dn}(2Ku) &= \frac{\left(1-2q^2\right)+2q\left(1-q^2\right)\cos 2\pi u}{\left(1+2q^2\right)-2q\left(1+q^2\right)\cos 2\pi u} + O\left(k^8\right) \; .
\end{aligned}\right\} \qquad (25)$$

Consequently, these along with Equation 21 substituted into Equations 19a and 19b provide formulae for $\cos\lambda_1$ and $\cosh\lambda_2$ with errors of the eighth order in k_1 and k_2. The resulting formulae are not written out here.

The coordinates and momenta of the intermediate orbit are expressed in terms of the parameter s_1. To locate position in orbit for a given instant it is necessary to know the relationship between s_1 and t. Under differentiation Equations 13a and 13b become

$$\left.\begin{aligned}
dt &= -\frac{1}{4}\,\frac{\cos^2\lambda_1\, d\lambda_1}{\pm\sqrt{\mu_0+\mu_1\cos\lambda_1+\mu_2\cos^2\lambda_1}} + \frac{1}{4}\,\frac{\cosh^2\lambda_2\, d\lambda_2}{\pm\sqrt{\nu_0+\nu_1\cosh\lambda_2+\nu_2\cosh^2\lambda_2}} \; , \\[2ex]
0 &= -\frac{d\lambda_1}{\pm\sqrt{\mu_0+\mu_1\cos\lambda_1+\mu_2\cos^2\lambda_1}} + \frac{d\lambda_2}{\pm\sqrt{\nu_0+\nu_1\cosh\lambda_2+\nu_2\cosh^2\lambda_2}} \; .
\end{aligned}\right\}$$

Also, from Equation 16, we find

$$ds_1 = \frac{d\lambda_1}{\pm\sqrt{\mu_0+\mu_1\cos\lambda_1+\mu_2\cos^2\lambda_1}} \; , \qquad ds_2 = ds_1 \; .$$

Thus,

$$dt = \frac{1}{4}\left(\cosh^2\lambda_2 - \cos^2\lambda_1\right) ds_1$$

or

$$
\left.\begin{aligned}
t - \beta_1 &= \frac{1}{4}\int_0^{s_1} \left(\cosh^2\hat{\lambda}_2 - \cos^2\hat{\lambda}_1\right) du \ , \\
\cosh\hat{\lambda}_2 &= \frac{2r_1 - (r_1 - 1)\ \mathrm{sn}^2 a_2 (u - \beta_2)}{2 + (r_1 - 1)\ \mathrm{sn}^2 a_2 (u - \beta_2)} \ , \\
\cos\hat{\lambda}_1 &= \frac{\rho_2 + 1 - 2\rho_2\ \mathrm{sn}^2 a_1 u}{\rho_2 + 1 - 2\ \mathrm{sn}^2 a_1 u} \ .
\end{aligned}\right\} \quad (26)
$$

There appears to be no very simple compact expression devoid of integrals for t in terms of s_1. However, we can get the lead-off terms in an expansion in powers of k_1 and k_2 by using Equations 24a through 24f and effecting the integration. The resulting formula is

$$
\begin{aligned}
4(t - \beta_1) = {} & \left[1 - \rho_2^2 + \frac{\rho_2(1-\rho_2)^2}{2} k_1^2 - 2\frac{r_1 + 1}{r_1 - 1} k_2^2\right] s_1 \\
& + \left[\frac{2}{b_1}\rho_2\sqrt{\rho_2^2 - 1} + \frac{1}{b_1}\frac{(2\rho_2^2 - 1)\sqrt{\rho_2^2 - 1}}{2} k_1^2\right] \arctan\left(\sqrt{\frac{\rho_2 - 1}{\rho_2 + 1}} \tan\frac{b_1 s_1}{2}\right) \\
& + \left[\frac{2(r_1 - 1)}{b_2}\sqrt{\frac{r_1 + 1}{2}} + \frac{2(r_1 + 1)}{b_2(r_1 - 1)} \cdot \frac{7r_1^2 + 14r_1 + 27}{4\sqrt{2}\sqrt{r_1 + 1}} k_2^2\right] \arctan\left(\sqrt{\frac{r_1 + 1}{2}} \tan\frac{b_2(s_1 - \beta_2)}{2}\right) \\
& + \frac{\rho_2^2 - 1}{b_1}\frac{\sin b_1 s_1}{\rho_2 + \cos b_1 s_1} + \frac{\rho_2^2 - 1}{4b_1} \cdot \frac{\sin b_1 s_1 \cdot (2\rho_2^2 + 1 + 3\rho_2\cos b_1 s_1)}{(\rho_2 + \cos b_1 s_1)^2} k_1^2 \\
& + \frac{2(r_1^2 - 1)}{b_2} \cdot \frac{\sin b_2(s_1 - \beta_2)}{(r_1 + 3) - (r_1 - 1)\cos b_2(s_1 - \beta_2)} - \frac{2(r_1 + 1)}{b_2(r_1 - 1)} \cdot \frac{(r_1 - 1)\sin b_2(s_1 - \beta_2) \cdot \left[r_1^2 + 2r_1 + 13 + (-r_1^2 - 6r_1 + 7)\cos b_2(s_1 - \beta_2)\right]}{4\left[(r_1 - 1)\cos b_2(s_1 - \beta_2) - (r_1 + 3)\right]^2} k_2^2 \\
& + \left[\frac{2(r_1 - 1)}{b_2}\sqrt{\frac{(r_1 + 1)}{2}} + \frac{2(r_1 + 1)}{b_2(r_1 - 1)} \cdot \frac{7r_1^2 + 14r_1 + 27}{4\sqrt{2}\sqrt{r_1 + 1}} k_2^2\right] \arctan\left(\sqrt{\frac{r_1 + 1}{2}} \tan\frac{b_2\beta_2}{2}\right) \\
& + \frac{2(r_1^2 - 1)}{b_2} \cdot \frac{\sin b_2\beta_2}{(r_1 + 3) - (r_1 - 1)\cos b_2\beta_2} - \frac{2(r_1 + 1)}{b_2(r_1 - 1)} \cdot \frac{(r_1 - 1)\sin b_2\beta_2 \cdot \left[r_1^2 + 2r_1 + 13 + (-r_1^2 - 6r_1 + 7)\cos b_2\beta_2\right]}{4\left[(r_1 - 1)\cos b_2\beta_2 - (r_1 + 3)\right]^2} k_2^2 + \cdots ,
\end{aligned} \quad (27)
$$

where

$$
b_1 = \frac{\pi a_1}{K(k_1)} \ , \qquad b_2 = \frac{\pi a_2}{K(k_2)} \ .
$$

The last two lines of Equation 27 represent a constant such that the entire expression vanishes with ϑ_1, in accordance with Equation 26. Note that we have taken the constants of integration in such a way that $\lambda_1 = 0$ and $t = \beta_1$ when $\vartheta_1 = 0$.

Equation 27 is not useful if k_2 is small because it is $r_1 - 1$ that is small, since the integrations have introduced $r_1 - 1$ in the denominators. This difficulty can be avoided in this case by expressing the formula for $\cosh \lambda_2$ in Equation 24b in terms of k_2 and r_2 (instead of k_2 and r_1) before substituting in Equation 26 and integrating.

PERTURBATION EQUATIONS

The variation equations are formed on the Hamiltonian $\widetilde{H}_1$ given in Equation 7:

$$\widetilde{H}_1 = -\left(\cosh^2 \lambda_2 - \cos^2 \lambda_1\right)^{-1} \left(\left\{\sinh \lambda_2 \cdot \left[\cosh \lambda_2 + (1 - 2\mu) \cos \lambda_1\right] - u(\lambda_1)\right\} \left\{\Lambda_1 + \frac{1}{2} u(\lambda_1)\right\} \right.$$
$$\left. + \left\{\sin \lambda_1 \cdot \left[(1 - 2\mu) \cosh \lambda_2 + \cos \lambda_1\right] - v(\lambda_2)\right\} \left\{\Lambda_2 + \frac{1}{2} v(\lambda_2)\right\}\right) .$$

A little elliptic function manipulation on Equations 17a and 17b gives formulae for the trigonometric and hyperbolic sines which occur in $\widetilde{H}_1$:

$$\left.\begin{aligned} \sin \lambda_1 &= \frac{2\sqrt{\rho_2^2 - 1}\, \operatorname{sn} a_1 \vartheta_1 \operatorname{cn} a_1 \vartheta_1}{\rho_2 - 1 + 2 \operatorname{cn}^2 a_1 \vartheta_1}, \\ \sinh \lambda_2 &= \frac{2\sqrt{r_1^2 - 1}\, \operatorname{cn} a_2 \vartheta_2}{r_1 + 1 - (r_1 - 1) \operatorname{cn}^2 a_2 \vartheta_2} . \end{aligned}\right\}$$

Similarly,

$$\frac{1}{2} u(\lambda_1) = \pm \sqrt{u_0 + u_1 \cos \lambda_1 + u_2 \cos^2 \lambda_1}$$
$$= \pm \frac{\sqrt{(u_0 + u_1 + u_2)(\rho_2 + 1)^2 - 2(\rho_2 + 1)\left[2u_0 + u_1(\rho_2 + 1) + 2u_2 \rho_2\right] \operatorname{sn}^2 a_1 \vartheta_1 + 4(u_0 + u_1 \rho_2 + u_2 \rho_2^2) \operatorname{sn}^4 a_1 \vartheta_1}}{\rho_2 + 1 - 2 \operatorname{sn}^2 a_1 \vartheta_1},$$

and

$$\frac{1}{2} v(\lambda_2) = \pm \sqrt{v_0 + v_1 \cosh \lambda_2 + v_2 \cosh^2 \lambda_2}$$
$$= \pm \frac{\sqrt{4(v_0 + v_1 r_1 + v_2 r_1^2) + 2(r_1 - 1)\left[2v_0 + v_1(r_1 - 1) - 2v_2 r_1\right] \operatorname{sn}^2 a_2 \vartheta_2 + (r_1 - 1)^2 (v_0 - v_1 + v_2) \operatorname{sn}^4 a_2 \vartheta_2}}{2 + (r_1 - 1) \operatorname{sn}^2 a_2 \vartheta_2} .$$

The constants u_i, v_i are still at our disposal. For instance, the radicands in $u(\lambda_1)$ and $v(\lambda_2)$ are perfect squares if we take $u_1^2 - 4u_0 u_2 = v_1^2 - 4v_0 v_2 = 0$, and the formulae are somewhat simplified.

The equations of variation for the canonical constants are of the form

$$\left.\begin{aligned} \frac{dh}{dt} &= \frac{\partial \hat{H}_1}{\partial \beta_1} , \\ \frac{d\alpha}{dt} &= \frac{\partial \hat{H}_1}{\partial \beta_2} , \end{aligned}\right\} \qquad \left.\begin{aligned} \frac{d\beta_1}{dt} &= -\frac{\partial \hat{H}_1}{\partial h} , \\ \frac{d\beta_2}{dt} &= -\frac{\partial \hat{H}_1}{\partial \alpha} . \end{aligned}\right\} \tag{28}$$

We may think of the coordinates and momenta as being expressed in terms of the parameter s_1 so that $\hat{H}_1$ in Equation 28 is to be regarded as known at this point as a function of s_1 rather than of t. Consequently, we transform Equation 28 to get s_1 as the independent variable in place of t. Let us write $h = \alpha_1$, $\alpha = \alpha_2$. Then, with the independent variable's being t the variation equations are

$$\left.\begin{aligned} \frac{d\alpha_i}{dt} &= \frac{\partial}{\partial \beta_i} \hat{H}_1\left[\lambda_r\left(\alpha_i, \beta_i, t\right), \Lambda_r\left(\alpha_i, \beta_i, t\right)\right] , \\ \frac{d\beta_i}{dt} &= -\frac{\partial}{\partial \alpha_i} \hat{H}_1\left[\lambda_r\left(\alpha_i, \beta_i, t\right), \Lambda_r\left(\alpha_i, \beta_i, t\right)\right] . \end{aligned}\right\} \tag{29}$$

The Hamiltonian for the problem we are treating is independent of any explicit value of t so we shall work out the transformation to s_1 of Equation 29 under this assumption. Let us represent the relation between s_1 and t by

$$\left.\begin{aligned} t &= \phi\left(\alpha_i, \beta_i, s_1\right) , \\ s_1 &= \psi\left(\alpha_i, \beta_i, t\right) . \end{aligned}\right\} \tag{30}$$

In our particular case this transformation is, of course, that of Equation 26. The total differential of t is, according to Equation 30,

$$dt = \frac{\partial \phi}{\partial \alpha_r} d\alpha_r + \frac{\partial \phi}{\partial \beta_r} d\beta_r + \frac{\partial \phi}{\partial s_1} ds_1$$

in which the summation convention is observed, i.e., a term having an index repeated is to be regarded as a sum taken over that index. Thus, the variational equations take on the form

$$\frac{d\alpha_i}{ds_1} = \delta_{ir} \frac{d\alpha_r}{ds_1} = \frac{\partial \hat{H}_1}{\partial \beta_i} \cdot \left[\frac{\partial \phi}{\partial \alpha_r} \frac{d\alpha_r}{ds_1} + \frac{\partial \phi}{\partial \beta_r} \frac{d\beta_r}{ds_1} + \frac{\partial \phi}{\partial s_1}\right] ,$$

$$\frac{d\beta_i}{ds_1} = \delta_{ir} \frac{d\beta_r}{ds_1} = -\frac{\partial \hat{H}_1}{\partial \alpha_i} \cdot \left[\frac{\partial \phi}{\partial \alpha_r} \frac{d\alpha_r}{ds_1} + \frac{\partial \phi}{\partial \beta_r} \frac{d\beta_r}{ds_1} + \frac{\partial \phi}{\partial s_1}\right],$$

in which δ_{ir} is the Kronecker delta: $\delta_{ir} = 0$, $i \neq r$, and $\delta_{ir} = 1$, $i = r$. In a standardized form the equations are

$$\left.\begin{aligned} \left(\frac{\partial \hat{H}_1}{\partial \beta_i}\frac{\partial \phi}{\partial \alpha_r} - \delta_{ir}\right)\frac{d\alpha_r}{d\sigma_1} + \frac{\partial \hat{H}_1}{\partial \beta_i}\frac{\partial \phi}{\partial \beta_r}\frac{d\beta_r}{d\sigma_1} &= -\frac{\partial \hat{H}_1}{\partial \beta_i}\frac{\partial \phi}{\partial \sigma_1}, \\ -\frac{\partial \hat{H}_1}{\partial \alpha_i}\frac{\partial \phi}{\partial \alpha_r}\frac{d\alpha_r}{d\sigma_1} + \left(-\frac{\partial \hat{H}_1}{\partial \alpha_i}\frac{\partial \phi}{\partial \beta_r} - \delta_{ir}\right)\frac{d\beta_r}{d\sigma_1} &= \frac{\partial \hat{H}_1}{\partial \alpha_i}\frac{\partial \phi}{\partial \sigma_1}, \end{aligned}\right\} \tag{31}$$

where

$$\alpha_1 = h, \qquad \alpha_2 = \alpha,$$

wherein the summation convention applies. It remains to be seen how the various quantities in Equation 31 are to be computed. Now

$$\begin{aligned} \hat{H}_1 \equiv \hat{H}_1\left[\lambda_r(\alpha_i, \beta_i, t), \Lambda_r(\alpha_i, \beta_i, t)\right] &= \hat{H}_1\left[\lambda_r\left(\alpha_i, \beta_i, \phi(\alpha_i, \beta_i, \sigma_1)\right), \Lambda_r\left(\alpha_i, \beta_i, \phi(\alpha_i, \beta_i, \sigma_1)\right)\right] \\ &\equiv \tilde{H}_1(\alpha_r, \beta_r, \sigma_1) = \tilde{H}_1\left(\alpha_r, \beta_r, \psi(\alpha_r, \beta_r, t)\right) \end{aligned}$$

and it is this function $\tilde{H}_1(\alpha_r, \beta_r, \sigma_1)$ that the $\tilde{H}_1$ in Equation 7 actually represents. The $\partial \hat{H}_1/\partial \beta_i$ and $\partial \hat{H}_1/\partial \alpha_i$ that occur in the coefficients in Equation 31 are thus found from

$$\left.\begin{aligned} \frac{\partial \hat{H}_1}{\partial \beta_i} &= \frac{\partial \tilde{H}_1}{\partial \beta_i} + \frac{\partial \tilde{H}_1}{\partial \sigma_1}\frac{\partial \psi}{\partial \beta_i}, \\ \frac{\partial \hat{H}_1}{\partial \alpha_i} &= \frac{\partial \tilde{H}_1}{\partial \alpha_i} + \frac{\partial \tilde{H}_1}{\partial \sigma_1}\frac{\partial \psi}{\partial \alpha_i}, \end{aligned}\right\} \tag{32}$$

with t then replaced by $\phi(\alpha_r, \beta_r, \sigma_1)$. As a matter of fact, $\partial\psi/\partial\beta_i$ and $\partial\psi/\partial\alpha_i$ will be computed in practice by implicit differentiation of $t = \phi(\alpha_i, \beta_i, \sigma_1)$ where $\sigma_1 \equiv \psi$, which automatically replaces t by (a function of) σ_1. The derivatives $\partial\phi/\partial\alpha_r$, $\partial\phi/\partial\beta_r$, $\partial\phi/\partial\sigma_1$ are simply the straight partial derivatives of the function $\phi(\alpha_i, \beta_i, \sigma_1)$.

We shall not carry out the details of determining explicitly the coefficients in Equation 31 for our particular case but shall content ourselves with working out necessary auxiliary formulae. We find that

$$\left.\begin{aligned} \left.\begin{aligned} \frac{\partial \mu_2}{\partial h} &= -\frac{1}{2}, \\ \frac{\partial \mu_0}{\partial \alpha} &= -1, \end{aligned}\right\} \quad \frac{\partial \nu_2}{\partial h} &= \frac{1}{2}, \\ \frac{\partial \nu_0}{\partial \alpha} &= 1, \end{aligned}\right\} \tag{33}$$

with all remaining derivatives of μ_i, ν_i with respect to α, k, and β_i zero. Furthermore,

$$\left.\begin{aligned}
\frac{\partial \rho_1}{\partial k} &= \frac{\rho_1^2}{4a_1^2 k_1^2}, & \frac{\partial \rho_1}{\partial \alpha} &= \frac{1}{2a_1^2 k_1^2}, & \frac{\partial r_1}{\partial k} &= \frac{r_1^2}{4a_2^2}, & \frac{\partial r_1}{\partial \alpha} &= \frac{1}{2a_2^2}, \\
\frac{\partial \rho_2}{\partial k} &= -\frac{\rho_2^2}{4a_1^2 k_1^2}, & \frac{\partial \rho_2}{\partial \alpha} &= -\frac{1}{2a_1^2 k_1^2}, & \frac{\partial r_2}{\partial k} &= -\frac{r_2^2}{4a_2^2}, & \frac{\partial r_2}{\partial \alpha} &= -\frac{1}{2a_2^2},
\end{aligned}\right\} \tag{34}$$

$$\frac{\partial \rho_i}{\partial \beta_j} = \frac{\partial r_i}{\partial \beta_j} = 0\,, \tag{35}$$

$$\left.\begin{aligned}
\frac{\partial a_1}{\partial k} &= \frac{\rho_1 - \rho_2 + 2\rho_1 \rho_2}{16a_1}, & \frac{\partial a_2}{\partial k} &= \frac{r_1 r_2}{4a_2 (r_1 - r_2)}, \\
\frac{\partial a_1}{\partial \alpha} &= \frac{2 - \rho_1 + \rho_2}{8a_1(\rho_1 - \rho_2)}, & \frac{\partial a_2}{\partial \alpha} &= \frac{1}{2a_2 (r_1 - r_2)},
\end{aligned}\right\} \tag{36}$$

$$\frac{\partial a_i}{\partial \beta_j} = 0\,, \tag{37}$$

$$\left.\begin{aligned}
\frac{\partial k_1}{\partial k} &= \frac{\left(-\rho_1^2 - \rho_2^2 + 2\rho_1^2 \rho_2^2\right) k_1}{16a_1^2 (\rho_1 - \rho_2)^2}, & \frac{\partial k_2}{\partial k} &= +\frac{r_1^2 + r_2^2 - 2r_1^2 r_2^2}{16a_2^2 (r_1 - r_2)^2 k_2}, \\
\frac{\partial k_1}{\partial \alpha} &= \frac{\left(\rho_1^2 + \rho_2^2 - 2\right) k_1}{8a_1^2 (\rho_1 - \rho_2)^2}, & \frac{\partial k_2}{\partial \alpha} &= -\frac{r_1^2 + r_2^2 - 2}{8a_2^2 (r_1 - r_2)^2 k_2},
\end{aligned}\right\} \tag{38}$$

$$\frac{\partial k_i}{\partial \beta_j} = 0\,. \tag{39}$$

As already noted, the determination of $\partial\psi/\partial k$, $\partial\psi/\partial\alpha$, $\partial\psi/\partial\beta_i$ will come from implicit differentiation of $t = \phi(\alpha, k, \beta_i, \vartheta_1)$, that is to say, implicit differentiation of Equation 26. The results are

$$\left.\begin{aligned}
\frac{\partial \psi}{\partial k} &= -\left(\cosh^2 \lambda_2 - \cos^2 \lambda_1\right)^{-1} \int_0^{\vartheta_1} \left(2 \cosh \hat{\lambda}_2 \frac{\partial \cosh \hat{\lambda}_2}{\partial k} - 2 \cos \hat{\lambda}_1 \frac{\partial \cos \hat{\lambda}_1}{\partial k}\right) du\,, \\
\frac{\partial \psi}{\partial \alpha} &= -\left(\cosh^2 \lambda_2 - \cos^2 \lambda_1\right)^{-1} \int_0^{\vartheta_1} \left(2 \cosh \hat{\lambda}_2 \frac{\partial \cosh \hat{\lambda}_2}{\partial \alpha} - 2 \cos \hat{\lambda}_1 \frac{\partial \cos \hat{\lambda}_1}{\partial \alpha}\right) du\,, \\
\frac{\partial \psi}{\partial \beta_1} &= -\frac{4}{\cosh^2 \lambda_2 - \cos^2 \lambda_1}\,, \\
\frac{\partial \psi}{\partial \beta_2} &= \left(\cosh^2 \lambda_2 - \cos^2 \lambda_1\right)^{-1} \left\{\left[\frac{2r_1 - (r_1 - 1)\, \mathrm{sn}^2\, a_2 (\vartheta_1 - \beta_2)}{2 + (r_1 - 1)\, \mathrm{sn}^2\, a_2 (\vartheta_1 - \beta_2)}\right]^2 - \left[\frac{2r_1 - (r_1 - 1)\, \mathrm{sn}^2\, a_2 \beta_2}{2 + (r_1 - 1)\, \mathrm{sn}^2\, a_2 \beta_2}\right]^2\right\}\,.
\end{aligned}\right\} \tag{40}$$

$$\left.\begin{aligned}
\frac{\partial \cosh \hat{\lambda}_2}{\partial k} &= \left[2+(r_1-1)\,\mathrm{sn}^2\,a_2(u-\beta_2)\right]^{-2}\left\{\frac{r_1^2}{a_2^2}\,\mathrm{cn}^2\,a_2(u-\beta_2) - \frac{r_1\,r_2\,(r_1^2-1)}{a_2\,(r_1-r_2)}\,(u-\beta_2)\,\mathrm{sn}\,a_2(u\right.\\
&\qquad \left. -\beta_2)\,\mathrm{cn}\,a_2(u-\beta_2)\,\mathrm{dn}\,a_2(u-\beta_2) - 4\,(r_1^2-1)\,\mathrm{sn}\,a_2(u-\beta_2)\,\frac{\partial\,\mathrm{sn}\,a_2(u-\beta_2)}{\partial k_2}\,\frac{\partial k_2}{\partial k}\right\}\,,\\
\frac{\partial \cosh \hat{\lambda}_2}{\partial \alpha} &= \left[2+(r_1-1)\,\mathrm{sn}^2\,a_2(u-\beta_2)\right]^{-2}\left\{\frac{2}{a_2^2}\,\mathrm{cn}^2\,a_2(u-\beta_2) - \frac{2\,(r_1^2-1)}{a_2\,(r_1-r_2)}\,(u-\beta_2)\,\mathrm{sn}\,a_2(u\right.\\
&\qquad \left. -\beta_2)\,\mathrm{cn}\,a_2(u-\beta_2)\,\mathrm{dn}\,a_2(u-\beta_2) - 4\,(r_1^2-1)\,\mathrm{sn}\,a_2(u-\beta_2)\,\frac{\partial\,\mathrm{sn}\,a_2(u-\beta_2)}{\partial k_2}\,\frac{\partial k_2}{\partial \alpha}\right\}\,,\\
\frac{\partial \cos \hat{\lambda}_1}{\partial k} &= \left[(\rho_2+1) - 2\,\mathrm{sn}^2\,a_1\,u\right]^{-2}\left\{\frac{\rho_2^2}{a_1^2\,k_1^2}\,\mathrm{sn}^2\,a_1\,u\,\mathrm{cn}^2\,a_1\,u\right.\\
&\qquad \left. - \frac{(\rho_2^2-1)\,(\rho_1-\rho_2+2\rho_1\,\rho_2)}{4a_1}\,u\,\mathrm{sn}\,a_1\,u\,\mathrm{cn}\,a_1\,u\,\mathrm{dn}\,a_1\,u - 4(\rho_2^2-1)\,\mathrm{sn}\,a_1\,u\,\frac{\partial\,\mathrm{sn}\,a_1\,u}{\partial k_1}\,\frac{\partial k_1}{\partial k}\right\}\,,\\
\frac{\partial \cos \hat{\lambda}_1}{\partial \alpha} &= \left[(\rho_2+1) - 2\,\mathrm{sn}^2\,a_1\,u\right]^{-2}\left\{\frac{2}{a_1^2\,k_1^2}\,\mathrm{sn}^2\,a_1\,u\,\mathrm{cn}^2\,a_1\,u\right.\\
&\qquad \left. - \frac{(\rho_2^2-1)\,(\rho_1-\rho_2+2\rho_1\,\rho_2)}{2a_1\,(\rho_1-\rho_2)}\,u\,\mathrm{sn}\,a_1\,u\,\mathrm{cn}\,a_1\,u\,\mathrm{dn}\,a_1\,u - 4(\rho_2^2-1)\,\mathrm{sn}\,a_1\,u\,\frac{\partial\,\mathrm{sn}\,a_1\,u}{\partial k_1}\,\frac{\partial k_1}{\partial \alpha}\right\}\,.
\end{aligned}\right\}\quad (41)$$

CONCLUDING REMARKS

The formulae of the preceding section are lengthy and involved, and it appears difficult to get pertinent information out of Equation 31. For a first order theory it may well be better to deal with the variation equations in t of Equation 28, rather than those in ϕ_1 of Equation 31. The usefulness of the partitioning introduced in this paper will, of course, depend largely on how much trouble it is to handle the perturbation equations and on how large the perturbations turn out to be. As far as application to the Apollo project is concerned, work remains to be done to ascertain the probable range of values for r_i, ρ_i for these orbits, so that it can be determined whether the type of approximation used in representing the intermediate orbit (i.e., low order in k_1 and k_2) will be reasonable.

A certain amount of latitude is present due to the arbitrary character of the functions $u(\lambda_1)$ and $v(\lambda_2)$ introduced at the partitioning of the original Hamiltonian. This paper considers a specific choice for the form of these functions, although six arbitrary constants are left available. An investigation could well be made concerning the feasibility and usefulness of some other choice for these functions.

REFERENCES

1. Wintner, A., "The Analytical Foundations of Celestial Mechanics," Princeton, N. J.: Princeton University Press, 1947.

2. Charlier, C. V., "Die Mechanik des Himmels," Leipzig: Veit and Co., 1902.

3. Samter, H., "Das Zweizentren-Problem in der Störungstheorie," *Astron. Nachr.* 217(5193-5194):129-152, April 10, 1922.

4. Magnus, von W., and Oberhettinger, F., "Formeln und Sätze für die Speziellen Funktionen der Mathematischen Physik," Berlin: Springer-Verlag, 1948.

Appendix A

List of Symbols

$\left.\begin{matrix}\mu \\ 1 - \mu\end{matrix}\right\}$	Two finite masses in the restricted three-body problem
x, y	Barycentric coordinates for a system rotating with masses such that particle coordinates for μ and $1 - \mu$ are $(1 - \mu, 0)$ and $(-\mu, 0)$
L, U	Lagrangian and force functions for the restricted three-body problem
r_1, r_2	Bipolar coordinates representing distances of (x, y) from the points $(-\mu, 0)$ and $(1 - \mu, 0)$, respectively
ξ, η	Elliptic coordinates expressed in terms of r_1, r_2
$\mathfrak{p}_i$	Momenta conjugate to r_i
$\mathfrak{h}, \alpha$	Canonical constant in Equation 9
β_1, β_2	Canonical constants in Equation 10
μ_i, ν_i	Constants defined by Equation 12